Int. J. Heat Mass Transfer. Vol. 18, pp. 1213–1215. Pergamon Press 1975. Printed in Great Britain

SHORTER COMMUNICATIONS

UPPER AND LOWER BOUNDS TO THERMAL INSTABILITY CRITERIA FOR COMPLETELY CONFINED FLUIDS INSIDE ARBITRARY CONFIGURATIONS*

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(*Received* 10 *February* 1975)

NOMENCLATURE

B, boundary of region C;
C, region of container;
P, pressure perturbation;
Ra, Rayleigh number $Ra = g\alpha\beta h^4/k\nu$; where g, α, β, h, k, ν are gravity constant, coefficient of thermal expansion, temperature gradient, characteristic length, thermal diffusivity, and kinematic viscosity, respectively;
u, v, w, cartesian velocity components;
x, y, z, cartesian coordinates;
θ, temperature perturbation;
λ, eigenvalue $\lambda = Ra^{\frac{1}{2}}$.

INTRODUCTION

THE FIRST analytical investigations of thermal instability appears to be that of Lord Rayleigh [2], who was motivated by experiments made by Bernard [3, 4]. A survey of the earlier development in this field was presented by Ostrach [5], and by Chandrasekhar [6].

The solution of thermal instability problems depends strongly on the boundary conditions and on the boundary shape. Thus, separation of variables, which was applied successfully to infinite horizontal layers, by Rayleigh [2], by Jeffreys [7], by Pellew and Southwell [8], and by Chandrasekhar [6], cannot solve the completely confined case. Because of the mathematics involved some other configurations, which still had one infinite dimension, were then considered: the vertical infinite layer, by Ostrach [11] and by Yih [12]; the vertical infinite cylinder, by Yih [12]; the infinite horizontal cylinder by Weinbaum [13]; and the infinite channel, by Davies-Jones [14].

Entirely different methods were required to obtain the solutions for completely confined fluids: for the finite vertical circular cylinder (Charlson and Sani [15], [16]) and for the rectangular box (Davis [17], modified by Catton [18]). These last methods either directly rely on some variational formulation of the problem or are indirectly shown to be equivalent to such an approach.

The stationary properties of the critical Rayleigh number, and some variational formulations of the problem were already used for the infinite fluid layer, by Rayleigh [2], by Pellew and Southwell [8], and by Chandrasekhar [6], who also suggested an alternative formulation. However, the validity of a variational principle for the completely confined fluid has only recently been established, by Sani [19, 20, 15]; also see [21]. Once shown to exist, the variational principle may have several formulations, each convenient for some particular investigation. These formulations can be applied in two ways: they can lead to exact solutions, by the Rayleigh–Ritz method [15] or by the Galerkin method [17, 18]; They can also be used to obtain bounds to the critical Rayleigh number as shown by Sherman and Ostrach [22] and modified in [23].

This investigation utilizes the variational principle of Sani [19, 20, 15] in a modified form, together with some general properties of eliptic operators, to obtain both upper and lower bounds to the critical Rayleigh number for very general closed configurations. The method requires the knowledge of the detailed geometry, and is, therefore, less general than that of [23] for lower bounds; however, it yields better lower bounds (i.e. higher) than those obtained by [23], for many classes of configurations, and it also yields upper bounds.

ANALYSIS

(a) *Basic equations*

A convenient form of the non-dimensional, linearized, basic equation is [15, 17]:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} = 0 \tag{1}$$

$$\nabla^2 u - \frac{\partial P}{\partial x} = 0 \tag{2}$$

$$\nabla^2 v - \frac{\partial P}{\partial y} = 0 \tag{3}$$

$$\nabla^2 w - \frac{\partial P}{\partial z} + \lambda\theta = 0 \tag{4}$$

$$\nabla^2 \theta + \lambda w = 0 \tag{5}$$

with the boundary conditions

$$u = v = w = \theta = 0 \quad \text{on} \quad B \tag{6}$$

v, u, and w are the cartesian components of the velocity vector, and θ and P are the perturbations in the rest state temperature and pressure

$$\lambda = Ra^{\frac{1}{2}} \tag{7}$$

$$Ra = g\alpha\beta h^4/k\nu \quad \text{the Rayleigh number} \tag{8}$$

g, α, β, h, k, and ν are, respectively, the gravity accelerations, the coefficient of thermal expansion, the constant temperature gradient, the characteristic length, the thermal diffusivity, and the kinematic viscosity. The fluid is completely confined inside the region C (the container), with its rigid boundaries B. These boundaries have on them a point of maximum z and a point of minimum z (at least one point for each). The characteristic length h is defined

$$h = (\max z)_B - (\min z)_B. \tag{9}$$

*This research was supported by the U.S. Air Force Office of Scientific Research under Grant AFOSR NA 74-064.

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Table 1. Upper and lower bounds to *Ra* for vertical cylinders of flat hexagonal cross-section (Fig. 1)

b	0·125	0·25	0·50	0·50	1·00	1·00	2·00	2·00	3·00	4·00	6·00
b_0	0·25	0·50	0·50	1·00	0·50	2·00	2·00	3·00	4·00	4·00	6·00
b_i	0·125	0·25	0·25	0·50	0·25	1·00	1·00	1·00	3·00	3·00	5·00
Lower bound	1 554 480	115 596	48 178	14 615	14 615	5138	3773	2753	2337	2270	1991
Upper bound	9 802 960	638 754	115 596	48 179	64 271	6974	5138	5138	2558	2338	2008

Table 2. Upper and lower bounds to *Ra* for a cone (Fig. 2)

R	1·0	1·5	2·0	2·5	3·0	4·0	5·0	6·0	7·0	8·0	∞
Lower bound	2545	2009	1886	1810	1783	1749	1734	1726	1721	1718	1707
Upper bound	40 721	20 264	13 014	9839	8044	6245	5218	4550	4117	3810	1708
γ	0·50	0·61	0·67	0·72	0·75	0·82	0·99	1·03	1·07	1·09	

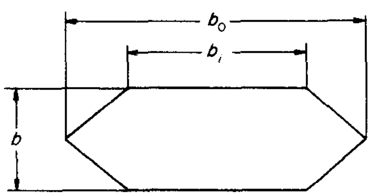

FIG. 1. Flat hexagonal cross-section.

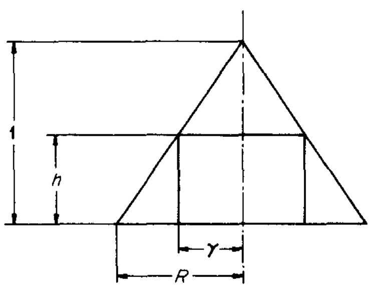

FIG. 2. Cone with inscribed cylinder.

The Boussinesq approximation [1] is assumed to hold, i.e. the properties of the fluid are assumed constant, including its density.

A variational formulation equivalent to equations (2) to (5) is (e.g. Charlson and Sani [15]):

$$\frac{1}{\lambda} = \max\left\{2\int_c \theta w \, dv \Big/ \int_c [\nabla u)^2 + (\nabla v)^2 + (\nabla w)^2 + (\nabla\theta)^2] \, dv\right\} \quad (10)$$

where u, v, w and θ must also satisfy equations (1) and (6).

(b) *Upper and lower bounds theorem*

Given two containers, one of which can be completely enclosed inside the other, the critical Rayleigh number of the inside container is an upper bound to that of the outside container, and the critical Rayleigh number of the outside container is a lower bound to that of the inside one.

This theorem, which expresses the fact that Rayleigh number is monotonic with respect to the domain, follows directly from a general property of elliptic eigenvalue problems, as used by Sani [24], by Joseph [25], and considered by Garabedian [26].

(c) *Natural boundary conditions*

The variational principle can be formally applied for cases where either $\theta = 0$ or both $\theta = 0$ and $u = v = w = 0$ are not imposed on parts of the boundary. Standard variational techniques yield either $\partial\theta/\partial n = 0$ or both $\partial\theta/\partial n = 0$ and $\partial u/\partial n = \partial v/\partial n = \partial w/\partial n = 0$ as natural boundary conditions on the corresponding parts of the boundaries. The λ values obtained for these cases would thus be *lower bounds* to the values obtained for identical configurations, for which equation (6) is satisfied everywhere (this point is already made by Charlson and Sani [15]).

In general these cases will not describe solutions to thermal instability problems. However, there are two particular classes of problems for which those natural boundary conditions are physically meaningful:

(1) The piecewise cylindrical insulating wall. Part of the wall of the container is generated by the horizontal motion of a vertical straight line, i.e. is defined by $f(x, y) = 0$. If this wall is insulating then $\partial\theta/\partial n = 0$ on this wall.
(2) The cellular cell in an infinite layer. Because of symmetry on the cell's boundaries $\partial\theta/\partial n = \partial u/\partial n = \partial v/\partial n = \partial w/\partial n = 0$.

Upper bounds to the critical Rayleigh number for these two classes can still be obtained by the Upper Bound Theorem, in two stages: an upper bound is first obtained by changing all boundary conditions to those of equation (6); then an upper bound to the upper bound is obtained by the theorem. Similarly, a known solution for a configuration with some natural boundary conditions can serve as a lower bound for a container of identical configurations, but with conducting walls.

It seems that no such general rule can be set for lower bounds; hence each configuration with some natural boundary conditions on parts of its boundaries must be treated as a particular case.

NUMERICAL EXAMPLES

(a) *Vertical cylinder with hexagonal cross-section*

Let the hexagon side be 1·155.

Upper Bound is obtained by the circular cylinder inscribed inside the hexagonal cylinder, i.e. of radius 0·8660 × 1·155 = 1·0. The results of Charlson and Sani [15] are used to obtain $Ra < 2545$.

Lower Bound could be obtained by the circular cylinder which circumscribes the hexagonal cylinder, i.e. of radius 1·155. Interpolation from the results of Charlson and Sani [15] yield $Ra > 2284$. Unfortunately, both [15] and [16] give upper bounds to the values of Ra for the circular cylinder, which cannot be used to generate the lower bound. Therefore, another method must be used: the exact solution for the hexagonal cell in the infinite layer (i.e. $\partial\theta/\partial n = \partial u/\partial n = \partial v/\partial n = \partial w/\partial n = 0$ on the cylindrical boundaries, all natural boundary conditions), interpolated from Chandrasekhar [6], is a lower bound: $Ra > 1817$. (The method of [23] yields $Ra > 413$.)

(b) *Vertical cylinder with flat hexagonal cross-section (Fig. 1)*

Let the rectangular dimensions of the hexagon be b, b_0 and b_i.

Upper Bound is obtained by the solution for the rectangular region of sides b and b_i.

Lower Bound corresponds to the solutions for the rectangular region of sides b and b_0.

Using Catton's [18] results, some numerical values are presented in Table 1.

(c) *The cone (Fig. 2)*

Consider the cone of radius 1.

Lower bound is obtained by the solution for the hexagonal cylinder of side $2/\sqrt{3}$, hence $Ra > 1785$, Chandrasekhar [6] ([23] yields a lower bound of 1570).

Upper bound is obtained by inserting a circular cylinder inside the cone. It is noted that for such a cylinder the height and the radius are related ($h+r = 1$ for this example). Furthermore, when the critical Rayleigh number for such a cylinder is obtained in terms of h (its height), it must be multiplied by $(1/h)^4$ to yield the number for the cone. Trying several possible cylinders, the following results were obtained

For $r = 0{\cdot}333$ and $h = 0{\cdot}666$, $Ra < 59\,400$

For $r = 0{\cdot}5$ and $h = 0{\cdot}5$, $Ra < 40\,721$

For $r = 0{\cdot}6$ and $h = 0{\cdot}6$, $Ra < 78\,500$

This $Ra < 40\,721$ is chosen as the best upper bound for this case.

This process is repeated for cones of different radii, and the results are summarized in Table 2.

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